Ch12. Hypothesis Testing.

- Two competing hypotheses H_0. H_1.

  $X \sim P_\theta \quad \theta \in \Omega$.

  $H_0: \theta \in \Omega_0 \quad$ vs. $\quad H_1: \theta \in \Omega_1$

  $\Omega = \Omega_0 \cup \Omega_1 \qquad \Omega_0 \cap \Omega_1 = \phi$.

  focus on the case where θ is univariate.

- Critical region S: when $X \in S$. accept H_1
  $X \notin S$ accept H_0.

- power function $\beta(\cdot)$: $\beta(\theta) = P_\theta(X \in S)$.

  The chance of rejecting H_0 as a function of $\theta \in \Omega$.

  We hope $\beta(\theta) = 0$ for $\theta \in \Omega_0 \quad \beta(\theta) = 1$ for $\theta \in \Omega_1$

  impossible!

- significance level α: $\alpha = \sup_{\theta \in \Omega_0} P_\theta(X \in S)$.

  the level α is the worst chance of falsely rejecting H_0

- test or critical function $\varphi \in [0,1]$.
  Given $X=x$. $\varphi(x)$ is the chance of rejecting H_0.

  $$\beta(\theta) = P_\theta(\text{reject } H_0) = E_\theta P_\theta(\text{reject } H_0 | X) = E_\theta \varphi(X)$$

- Simple vs. Simple.
  H_0. H_1 contain a single parameter θ_0. θ_1 resp.

  $$\alpha = E_0 \varphi = \int \varphi(x) p_0(x) d\mu(x)$$
  $$E_1 \varphi = \int \varphi(x) p_1(x) d\mu(x).$$

  maximizing $E_1\varphi$ among all tests φ with $E_0\varphi = \alpha$.

Prop 12.1. suppose $K \geq 0$. φ^* maximizes $E_1\varphi - K E_0\varphi$ among all critical functions and $E_0\varphi^* = \alpha$. Then φ^* maximizes $E_1\varphi$ over all φ with level at most α.

Pf. suppose $E_0\varphi \leq \alpha$. Then

$$\begin{aligned} E_1\varphi &\leq E_1\varphi - K E_0\varphi + K\alpha \\ &\leq E_1\varphi^* - K E_0\varphi^* + K\alpha \\ &= E_1\varphi^*. \end{aligned}$$

- maximize $E_1\varphi - k E_0\varphi$.

$$E_1\varphi - kE_0\varphi = \int [p_1(x) - kp_0(x)]\,\varphi(x)\,d\mu(x)$$

$$= \int_{p_1(x) > k p_0(x)} |p_1(x) - kp_0(x)|\,\varphi(x)\,d\mu(x) - \int_{p_1(x) < k p_0(x)} |p_1(x) - kp_0(x)|\,\varphi(x)\,d\mu(x)$$

$\varphi^* = 1$ when $p_1(x) > kp_0(x)$

$\varphi^* = 0$ when $p_1(x) < kp_0(x)$

likelihood ratio. $L(x) = \dfrac{p_1(x)}{p_0(x)}$.

$$\varphi^*(x) = \begin{cases} 1 & L(x) > k \\ \text{any} & L(x) = k \\ 0 & L(x) < k \end{cases}$$ Likelihood ratio test.

Prop 12.2 (Neyman-Pearson Lemma) Given any level $\alpha \in [0,1]$, there exists a likelihood ratio test φ_α with level α. and any likelihood ratio test with level α maximizes $E_1\varphi$ among all tests with level at most α.

Prop 12.3. Fix $\alpha \in [0,1]$. let κ be the critical value for a likelihood ratio test φ_α and $B = \{x : p_1(x) \neq \kappa p_0(x)\}$.

If φ^* maximizes $E_1\varphi$ among all tests with level at most α. then φ^* and φ_α must agree on B.

$\longrightarrow$ If a test is optimal, it must be a likelihood ratio test.

$\longrightarrow$ If $P_0 \neq P_1$. $E_1 \varphi_\alpha > \alpha$

ex. $X \sim p_\theta(x) = \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$

$H_0: \theta = 1$. vs. $H_1: \theta = \theta_1$. $\theta_1 > 1$: pre-specified

$\varphi = 1$ if $\dfrac{p_1(x)}{p_0(x)} = \dfrac{\theta_1 e^{-\theta_1 x}}{e^{-x}} > k$ $\quad X < \dfrac{\log \frac{\theta_1}{k}}{\theta_1 - 1} = k'$

$\varphi = 0$ if $X > k'$ $\quad \varphi =$ anything in $[0,1]$ if $X = k'$

$\alpha = P_0(X < k') = \int_0^{k'} e^{-x} dx = 1 - e^{-k'}$ $\quad k' = -\log(1-\alpha)$

$\varphi_\alpha(X) = \begin{cases} 1 & X < -\log(1-\alpha) \\ 0 & X > -\log(1-\alpha). \end{cases}$

1. does not depend on θ_1
2. if φ is any test with level α $E_{\theta_1}\varphi \le E_{\theta_1}\varphi_\alpha$ for $\theta_1 > 1$

Ex: $X \sim Bin(n=2, \theta)$.

$H_0: \theta = \frac{1}{2}$ vs. $H_1: \theta = \frac{3}{4}$.

$$L(X) = \frac{p_1(X)}{p_0(X)} = \frac{\binom{2}{X}\left(\frac{3}{4}\right)^X\left(\frac{1}{4}\right)^{2-X}}{\binom{2}{X}\left(\frac{1}{2}\right)^X\left(\frac{1}{2}\right)^{2-X}} = \frac{3^X}{4}$$

Under H_0

$$L(X) = \begin{cases} \frac{1}{4} & w.p. \ \frac{1}{4} \\ \frac{3}{4} & w.p \ \frac{1}{2} \\ \frac{9}{4} & w.p \ \frac{1}{4}. \end{cases}$$

If $K < \frac{9}{4}$. $L(2) = \frac{9}{4} > K$ $\varphi(2) = 1$

$E_0 \varphi(X) \geq \varphi(2) P_0(X=2) = \frac{1}{4}$

If $K > \frac{9}{4}$ $\varphi = 0$.

So $K = \frac{9}{4}$. $\varphi(0) = \varphi(1) = 0$.

$5\% = E_0 \varphi(X) = \frac{1}{4}\varphi(0) + \frac{1}{2}\varphi(1) + \frac{1}{4}\varphi(2) = \frac{1}{4}\varphi(2)$

$\varphi(2) = \frac{1}{5}$.

- Uniformly most powerful test: A test φ^* with level α,

$$E_\theta \varphi^* \geq E_\theta \varphi \quad \forall\, \theta \in \Omega_1.$$

$$H_0: \theta \leq \theta_0. \quad vs. \quad H_1: \theta > \theta_0.$$

- monotone likelihood ratio: there exists a statistic $T = T(x)$. s.t. when $\theta_1 < \theta_2$, $\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)}$ is a nondecreasing function of T.

$p_{\theta_1} \neq p_{\theta_2}$, whenever $\theta_1 \neq \theta_2$.

ex. $$p_\theta(x) = e^{\eta(\theta) T(x) - B(\theta)} h(x)$$

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = e^{[\eta(\theta_2) - \eta(\theta_1)] T(x) + B(\theta_1) - B(\theta_2)} \quad \nearrow \text{ in } T(x).$$

Thm 12.9. Suppose the family of densities has a monotone likelihood ratio. then: 1. $\varphi^*(x) = \begin{cases} 1 & T(x) > c \\ \gamma & T(x) = c \\ 0 & T(x) < c \end{cases}$ is UMP test for

$H_0: \theta \leq \theta_0$. vs. $H_1: \theta > \theta_0$. It has level $\alpha = E_{\theta_0} \varphi^*$. adjust c and γ.

2. The power function $\beta(\theta) = E_\theta \varphi^*$ — nondecreasing and strictly increasing when $\beta(\theta) \in (0,1)$.

3. if $\theta_1 < \theta_0$. φ^* minimizes $E_{\theta_1} \varphi$ among all tests with $E_{\theta_0}\varphi = \alpha = E_{\theta_0}\varphi^*$.

ex. $X_1 \cdots X_n \overset{iid}{\sim} Unif(0,\theta)$

$M(x) = \min\{X_1 \cdots X_n\} > 0 \qquad T(x) = \max\{X_1 \cdots X_n\} < \theta.$

$$p_\theta(x) = \begin{cases} \frac{1}{\theta^n} & M(x) > 0.\ T(x) < \theta \\ 0 & o.w. \end{cases}$$

Suppose $\theta_2 > \theta_1$. $M(x) > 0$. $T(x) < \theta_2$. Then

$$\frac{p_{\theta_2}(x)}{p_{\theta_1}(x)} = \begin{cases} \frac{\theta_1^n}{\theta_2^n} & T(x) < \theta_1 \\ +\infty & T(x) \geq \theta_1 \end{cases}$$

monotone likelihood ratios

Test $H_0: \theta \leq 1$. vs. $H_1: \theta > 1$.

$$\varphi = \begin{cases} 1 & T \geq c \\ 0 & o.w. \end{cases}$$

uniformly m p.

$P_1(T \geq c) = 1 - c^n \qquad c = (1-\alpha)^{\frac{1}{n}}$

power: $\beta_\varphi(\theta) = P_\theta(T \geq c) = \begin{cases} 0 & \theta < c \\ 1 - \frac{1-\alpha}{\theta^n} & \theta \geq c \end{cases}$

Competing one:

$$\tilde{\varphi} = \begin{cases} \alpha & T < 1 \\ 1 & T \geq 1 \end{cases}$$

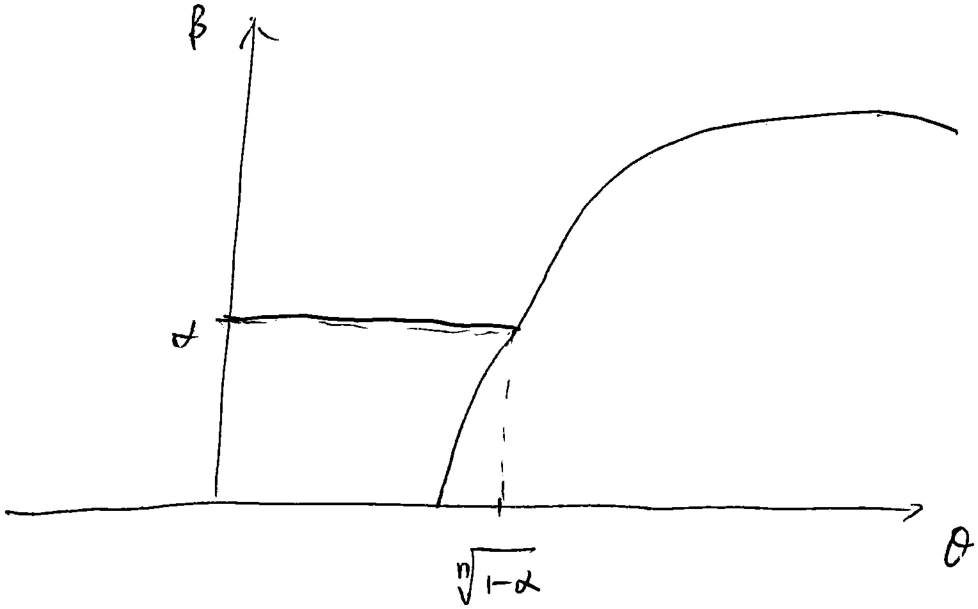
$E_\theta \tilde{\varphi} = \alpha$ for $\theta < 1$

For $\theta > 1$. $\beta_{\tilde{\varphi}}(\theta) = E_\theta \tilde{\varphi} = \alpha P_\theta(T < 1) + P_\theta(T \geq 1)$

$= \frac{\alpha}{\theta^n} + 1 - \frac{1}{\theta^n} = \beta_\varphi(\theta)$

~~The~~ power function for $\tilde{\varphi}$ and φ are the same under H_1

under H_0. the power function for φ is smaller than the power function for $\tilde{\varphi}$. $\longrightarrow$ φ is better.



β_φ and $\beta_{\tilde{\varphi}}$.

Duality Between Testing and Interval Estimation

- $S(X)$ — $1-\alpha$ confidence region for $\xi = \xi(\theta)$ if $P_\theta(\xi \in S(X)) \geq 1-\alpha$ for $\forall \theta \in \Omega$.

- For every ξ_0. Let $A(\xi_0)$ — acceptance region for $H_0: \xi(\theta) = \xi_0$ vs. $H_1: \xi(\theta) \neq \xi_0$.

$$P_\theta[X \in A(\xi(\theta))] \geq 1-\alpha \qquad \forall \theta \in \Omega.$$

- Define $S(x) = \{\xi : x \in A(\xi)\}$.

$\xi(\theta) \in S(X)$ iff $X \in A(\xi(\theta))$

$P_\theta(\xi(\theta) \in S(X)) = P_\theta(X \in A(\xi(\theta))) \geq 1-\alpha$.

$S(X)$ — $1-\alpha$ confidence region for ξ.

- Define $\varphi = \begin{cases} 1 & \xi_0 \notin S(X) \\ 0 & \text{o.w.} \end{cases}$

then if $\xi(\theta) = \xi_0$,

$$E_\theta \varphi = P_\theta(\xi_0 \notin S(X)) = P_\theta(\xi(\theta) \notin S(X)) \leq \alpha.$$

This test has level at most α. $H_0: \xi(\theta) = \xi_0$ vs. $H_1: \xi(\theta) \neq \xi_0$

$$\varphi = \begin{cases} 1 & \theta_0 \notin S(X) \\ 0 & \text{o.w.} \end{cases} \qquad \text{UMP} \quad H_0: \theta = \theta_0 \text{ vs. } H_1: \theta > \theta_0.$$

then for any $\theta > \theta_0$ $\quad E_\theta \varphi \geq E_\theta \varphi^*$.

$$P_\theta(\theta_0 \in S(X)) \leq P_\theta(\theta_0 \in S^*(X))$$

$\Rightarrow$ If θ is the true value, $S(X)$ has a smaller chance of containing any incorrect value $\theta_0 < \theta$.

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let $\Omega = (\underline{\omega}, \overline{\omega})$.

$$E_\theta \lambda\big(S(X) \cap (\underline{\omega}, \theta)\big) = E_\theta \int_{\underline{\omega}}^{\theta} I(\theta_0 \in S(X))\, d\theta_0$$

$$= \int \int_{\underline{\omega}}^{\theta} I(\theta_0 \in S(x))\, d\theta_0\, dP_\theta(x) = \int_{\underline{\omega}}^{\theta} P_\theta(\theta_0 \in S(X))\, d\theta_0$$

$$E_\theta \lambda\big(S^*(X) \cap (\underline{\omega}, \theta)\big) = \int_{\underline{\omega}}^{\theta} P_\theta(\theta_0 \in S^*(X))\, d\theta_0$$

$$E_\theta \lambda\big(S(X) \cap (\underline{\omega}, \theta)\big) \leq E_\theta \lambda\big(S^*(X) \cap (\underline{\omega}, \theta)\big).$$

Two-sided Hypotheses

- $H_0: \theta = \theta_0$. vs. $H_1: \theta \neq \theta_0$.

One parameter exponential family.

Thm 12.17. T — sufficient $\mathcal{P} = \{P_\theta : \theta \in \Omega\}$. Then for any test $\varphi = \varphi(X)$. the test $\psi = \psi(T) = E_\theta(\varphi(X) | T)$ has the same power function as φ, $E_\theta \psi(T) = E_\theta \varphi(X)$.

Lemma 12.18. $P_1 << P_0$. $\frac{dP_1}{dP_0} = L(x) = \frac{p_1(x)}{p_0(x)}$

$E_1 h(X) = E_0 h(X) L(X)$. $P_1(B) = E_0 1_B(X) L(X)$.

Thm 12.19 $X \sim p_\theta(x) = h(x) e^{\eta(\theta) \cdot T(x) - B(\theta)}$ $\theta \in \Omega$

$\Rightarrow$ $T = T(x)$ has density $q_\theta(t) = e^{\eta(\theta) \cdot t - B(\theta)}$ $\theta \in \Omega)$

Consider $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_1$.

$X \sim h(x) e^{\eta(\theta) T(X) - B(\theta)}$ $\theta \in \Omega$. $\eta \nearrow$ and diffentiable

$$\varphi_+ = \begin{cases} 1 & T > C_+ \\ \gamma_+ & T = C_+ \\ 0 & T < C_+ \end{cases} \qquad \varphi_- = \begin{cases} 1 & T < C_- \\ \gamma_- & T = C_- \\ 0 & T > C_- \end{cases}$$

These tests are most powerful for one-sided alternatives.

if $\theta_- < \theta_0 < \theta_+$, then φ_+ will have maximal power at θ_+, and φ_- will have maximal power at θ_-.

Thm 12.20. if η differentiable at θ and θ lies in the interior of Ω. then
$$\beta'(\theta) = \eta'(\theta) E_\theta T\varphi - B'(\theta)\beta(\theta).$$

pf. $$\beta'(\theta) = \frac{\partial}{\partial\theta}\int \varphi(x)\, e^{\eta(\theta)T(x)-B(\theta)}\, h(x)\, d\mu(x)$$
$$= \int \varphi(x)\,[\eta'(\theta)T(x) - B'(\theta)]\, e^{\eta(\theta)T(x)-B(\theta)}\, h(x)\, d\mu(x)$$

- Because φ_+ has maximal power for $\theta > \theta_0$.

$$\beta_\varphi'(\theta_0) = \lim_{\varepsilon\downarrow 0} \frac{\beta_\varphi(\theta_0+\varepsilon) - \beta_\varphi(\theta_0)}{\varepsilon} \le \lim_{\varepsilon\downarrow 0} \frac{\beta_{\varphi_+}(\theta_0+\varepsilon) - \beta_{\varphi_+}(\theta_0)}{\varepsilon} = \beta_{\varphi_+}'(\theta_0) \triangleq m_+$$

$$\beta_\varphi'(\theta_0) \geq \beta_{\varphi_-}'(\theta_0) \triangleq m_-.$$

For $m \in [m_-, m_+]$, let C_m denote the class of all level α tests with $\beta_\varphi'(\theta_0) = m$

- Def. $\cdot$ φ is called <u>two-sided</u> if there are finite constants $t_1 \le t_2$. s.t. $\varphi = \begin{cases} 1 & T < t_1 \text{ or } T > t_2 \\ 0 & T \in (t_1, t_2). \end{cases}$

In addition, the test should not be one-sided.

$E\varphi I(T \geq t_2)$ and $E\varphi I(T \leq t_1)$ should both be positive.

Thm 12.22. θ_0 — interior of Ω. $\alpha \in (0,1)$ $X - P_\theta(x)$.

η — differentiable $\nearrow$. $0 < \eta'(\theta_0) < \infty$, then for $m \in (m_-, m_+)$

there is a two-sided test $\varphi^* \in \mathcal{C}_m$. Any such test is

uniformly m.p. in class $\mathcal{C}_m$: For any $\varphi \in \mathcal{C}_m$.

$$E_\theta \varphi \leq E_\theta \varphi^*. \quad \forall \theta \in \Omega.$$

Unbiased Tests.

- A test φ for $H_0: \theta \in \Omega_0$ vs. $H_1: \theta \in \Omega_1$ with level α is unbiased if its power $\beta_\varphi(\theta) = E_\theta \varphi$ satisfies

$$\beta_\varphi(\theta) \le \alpha \quad \forall \theta \in \Omega_0$$
$$\beta_\varphi(\theta) \ge \alpha \quad \forall \theta \in \Omega_1$$

- if φ^* is UMP. it is automatically unbiased

$$\beta_{\varphi^*}(\theta) \ge \underbrace{\beta_\varphi(\theta)}_{=\alpha} \quad \text{for all } \theta \in \Omega_1$$

Thm 12.26. $\alpha \in (0,1)$. $\theta_0 \in$ interior of Ω. X has density $h(x) e^{\eta(\theta) T(x) - B(\theta)}$, η — differentiable. strictly increasing $0 < \eta'(\theta_0) < \infty$

then, there is a 2-sided, level α test φ^* with $\beta'_{\varphi^*}(\theta_0) = 0$. Any such test is UMP testing $H_0: \theta = \theta_0$, $H_1: \theta \ne \theta_0$ among all unbiased tests with level α.

Pf. θ_0 interior point. the power function for any unbiased test φ must have zero slope at θ_0. so $\varphi \in C_0 \to$ corollary Thm 12.22

provided $\theta \in (m_-, m_+)$.

ex. $X \sim p_\theta(x) = \theta e^{-\theta x} \quad x > 0$

$H_0: \theta = 1. \quad vs. \quad H_1: \theta \neq 1$

$$\varphi = \begin{cases} 0 & X \in (c_1, c_2) \\ 1 & X \le c_1 \text{ or } X \ge c_2 \end{cases}$$

φ is UMP if $\quad E_1\varphi = 1 - \int_{c_1}^{c_2} e^{-x}dx = 1 - e^{-c_1} + e^{-c_2} = \alpha$.

and $\quad E_1 X\varphi = E_1 X - E_1 X(1-\varphi) = 1 - \int_{c_1}^{c_2} x e^{-x} dx$

$= 1 - (1+c_1)e^{-c_1} + (1+c_2)e^{-c_2} = \alpha E_1 X = \alpha$.

$c_1 e^{-c_1} = c_2 e^{-c_2} \quad \Rightarrow \quad c_2 = -\lg(e^{-c_1} - 1 + \alpha)$

$c_1 e^{-c_1} = -(e^{-c_1} - 1 + \alpha)\lg(e^{-c_1} - 1 + \alpha)$

For $\alpha = 5\%$ $\quad c_1 = 0.042 \quad\quad c_2 = 4.765$

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or. $\quad P_1(X \le c_1) = P_1(X \ge c_2) = \frac{\alpha}{2}$. $\quad c_1 = -\lg(1 - \frac{\alpha}{2})$

$\quad c_2 = -\lg(\frac{\alpha}{2})$

$c_1 = 0.025 \quad c_2 = 3.689$.